

Nonextensivity and multifractality in low-dimensional dissipative systems

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Abstract

Power-law sensitivity to initial conditions at the edge of chaos provides a natural relation between the scaling properties of the dynamics attractor and its degree of nonextensivity as prescribed in the generalized statistics recently introduced by one of us (C.T.) and characterized by the entropic index q . We show that general scaling arguments imply that $1/(1-q) = 1/\alpha_{min} - 1/\alpha_{max}$, where α_{min} and α_{max} are the extremes of the multifractal singularity spectrum $f(\alpha)$ of the attractor. This relation is numerically checked to hold in standard one-dimensional dissipative maps. The above result sheds light on a long-standing puzzle concerning the relation between the entropic index q and the underlying microscopic dynamics.

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Nonextensivity is inherent to systems where long-range interactions or spatio-temporal complexity are present. Long-range forces are found at astrophysical as well as nanometric scales. Spatio-temporal complexity, a term introduced to describe the presence of long-range spatial and temporal correlations, is found in equilibrium statistical mechanics to emerge at critical points for second order phase transitions. Further, the concept of self-organized criticality has been recently introduced to describe driven systems which naturally evolve through transient non-critical states to a dynamical attractor poised at criticality [1]. Long-range spatial and temporal correlations are build up in these systems as a consequence of avalanche dynamics connecting metastable states. Self-organized criticality is conjectured to be in the origin of fractal structures, noise with a $1/f$ power-spectrum, anomalous diffusion, Lévy flights and punctuated equilibrium behavior [2], which are signatures of the nonextensive character of the dynamics attractor.

The proper statistical treatment of nonextensive systems seems to require a generalization of the Boltzmann-Gibbs-Shannon prescription based in the standard, extensive entropy $S = -\sum_i p_i \ln p_i$ (in units of Boltzmann constant), which provides a link between the microscopic dynamics and macroscopic thermodynamics. Inspired in the scaling properties of multifractals, one of us [3] has proposed a generalized nonextensive form of entropy

$$S_q = \frac{1 - \sum_i p_i^q}{q - 1} \quad \left(\sum_i p_i = 1; q \in \mathcal{R} \right), \quad (1)$$

which recovers the usual entropy form in the limit of $q \rightarrow 1$. The entropic index q controls the degree of nonextensivity reflected in the pseudo-additivity entropy rule $S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B)$, where A and B are two *independent* systems in the sense that the probabilities of $A+B$ *factorize* into those of A and of B . A wealth of works has been developed in the last few years showing that the above nonextensive thermostistical prescription retains much of the formal structure of the standard theory such as Legendre thermodynamic structure, H-theorem, Onsager reciprocity theorem, Kramers and Wannier relations and thermodynamic stability, among others [4]. Further, it has been applied to a series of nonextensive systems such as stellar polytropes [5], ferrofluids [6], two-dimensional

plasma turbulence [7], anomalous diffusion and Lévy flights [8], cosmology [9], peculiar velocities of galaxies [10] and inverse bremsstrahlung in plasma [11], among others [12].

In spite of these variety of applications of nonextensive thermostatics, a full and general understanding on the *precise* relation between the entropic index q and the underlying microscopic dynamics was still lacking. It has been conjectured that the generalized thermostatics is a natural frame for studying fractally structured systems [12] and simple relations were found between q and the characteristic exponents of anomalous diffusion and Lévy flights distributions [8]. Further, recent works have shown that the entropic index q has a monotonous dependence on the fractal dimension d_f of the dynamical chaotic attractor of dissipative nonlinear systems [13].

The aim of the present work is to develop the precise connection between the nonextensivity parameter q and the scaling properties of the critical attractor of nonlinear dynamical systems. Particularly, a prototype complex dynamical state will be taken to be the onset of chaos of nonlinear low-dimensional maps as it is well established that this state displays long-range temporal and spatial correlations. We will show that the *power-law* sensitivity to initial conditions at the edge of chaos provides a natural link between the entropic index q and the attractor's multifractal singularity spectrum. By using typical scaling properties of Feigenbaum-like maps we will show that $1/(1 - q) = 1/\alpha_{min} - 1/\alpha_{max}$, where α_{min} and α_{max} are the scaling exponents related respectively to the most concentrated and most rarefied portions of the attractor. We numerically illustrate the above result using standard one-dimensional maps such as generalized logistic-like maps and the circle map.

For the sake of simplicity, we will concentrate our attention to the simple case of one-dimensional non-linear dynamical systems. One of their most prominent features is related to the sensitivity to initial conditions. In order to quantify this aspect, Kolmogorov and Sinai's definition of the rate at which the amount of information about the initial conditions varies can be seen as :

$$K_1 \equiv \lim_{\tau \rightarrow 0} \lim_{l \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N\tau} [S_1(N) - S_1(0)], \quad (2)$$

where τ is a characteristic time step (in fact, $\tau \rightarrow 0$ for differential equations; $\tau = 1$ for discrete maps) and $S_1(0)$ and $S_1(N)$ stand, respectively, for the entropies evaluated at the times $t = 0$ and $t = N\tau$. The entropy can be evaluated by considering an ensemble of identical copies of the system and defining p_i as the fractional number of copies that are in the i -th cell (of size l) of the phase space. If one uses the extensive Boltzmann-Gibbs-Shannon entropy form $S = S_1 = -\sum_{i=1}^W p_i \ln p_i$, where W is the number of configurations at time t , the Kolmogorov-Sinai entropy results in

$$K_1 = \lim_{\tau \rightarrow 0} \lim_{l \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N\tau} \ln W(N)/W(0), \quad (3)$$

where equiprobability ($p_i = 1/W$) was assumed. Notice that the above expression implies in an exponential sensitivity to initial conditions $W(N) = W(0)e^{K_1 N\tau}$. K_1 plays (consistently with Pesin's equality) the role of the Liapunov exponent λ_1 which characterizes the exponential deviation of two initially nearby paths $\xi(t) \equiv \lim_{\Delta x(0) \rightarrow 0} \Delta x(t)/\Delta x(0) = e^{\lambda_1 t}$ ($\xi(t)$ is solution of $d\xi/dt = \lambda_1 \xi$). When $\lambda_1 < 0$ ($\lambda_1 > 0$) the system is said to be *strongly insensitive* (*strongly sensitive*) to the initial conditions. The marginal case of $\lambda_1 = 0$ occurs at the period doubling and tangent bifurcation points, as well as at the cumulating point of the period doubling bifurcations, i.e., at the threshold to chaos. The failure of the above scheme in distinguishing the sensitivity to initial conditions at these special points is related to the nonextensive (fractal-like) structure of their dynamical attractors.

Recently it was argued that, within the generalized nonextensive entropy of Eq. 1, the sensitivity to initial conditions of one-dimensional nonlinear maps becomes expressed as [13]

$$\xi(t) = [1 + (1 - q)\lambda_q t]^{1/(1-q)}, \quad (4)$$

which is solution of $d\xi/dt = \lambda_q \xi^q$. The above expression recovers the usual exponential sensitivity in the limite of $q \rightarrow 1$ (extensive statistics). Further, it implies a *power-law* sensitivity when nonextensivity takes place ($q \neq 1$). Previous numerical calculations have shown that the period doubling and tangent bifurcations exhibit *weak insensitivity* ($q > 1$) to initial conditions [13]. At the onset of chaos, *weak sensitivity* ($q < 1$) shows up and

the value of q was numerically verified to be closely related to the fractal dimension d_f of the dynamical attractor [13]. In what follows we will use scaling arguments to analytically express the entropic index q as a function of the fractal scaling properties of the attractor.

Actually, as in many nonlinear problems, the scaling behavior of the attractor is richer and more complex than is the case in usual critical phenomena. It is necessary to introduce a multifractal formalism in order to reveal its complete scaling behavior [14]. A central quantity in this formalism is the partition function $\chi_{\bar{q}}(N) = \sum_{i=1}^N p_i^{\bar{q}}$, where p_i represents the probability (integrated measure) on the i -th box among the N boxes of the measure (we use \bar{q} instead to the standard notation q in order to avoid confusion with the entropic index q). For example, in chaotic systems p_i is the fraction of times the trajectory visits the box i . In the $N \rightarrow \infty$ limit, the contribution to $\chi_{\bar{q}}(N) \propto N^{-\tau(\bar{q})}$, with a given \bar{q} , comes from a subset of all possible boxes, whose number scales with N as $N_{\bar{q}} \propto N^{f(\bar{q})}$, where $f(\bar{q})$ is the fractal dimension of the subset ($f(\bar{q} = 0)$ is the fractal dimension d_f of the support of the measure). The content on each contributing box is roughly constant and scales as $P_{\bar{q}} \propto N^{-\alpha(\bar{q})}$. These exponents are all related by a Legendre transformation $\tau(\bar{q}) = \bar{q}\alpha(\bar{q}) - f(\bar{q})$. The multifractal object is then characterized by the continuous function $f(\alpha)$, which reflects the different dimensions of the subsets with singularity strength α .

The multifractal formalism has been widely used to characterize the spectrum of singularities of some important objects arising in nonlinear dynamical systems as for example the critical cycle at the onset of chaos [14], diffusion limited aggregation, fully developed turbulence, random resistor networks, among many others [15]. The α values at the end points of the $f(\alpha)$ curve are the singularity strength associated with the regions in the set where the measure is most concentrated ($\alpha_{min} = \alpha(\bar{q} = +\infty)$) and most rarefied ($\alpha_{max} = \alpha(\bar{q} = -\infty)$).

The scaling properties of the most rarefied and most concentrated regions of multifractal dynamical attractors can be used to estimate the power-law divergence of nearby orbits. Consider the set of points in the attractor generated after a large number B of time steps ($p_i = 1/B$ is therefore the measure contained in each box). The most concentrated and most rarefied regions in the attractor are partitioned respectively in boxes of typical sizes

$l_{+\infty}$ and $l_{-\infty}$. From these, one may determine the end points of the singularity spectrum as $\alpha_{min} = \ln p_i / \ln l_{+\infty}$ (hence $l_{+\infty} \propto B^{-1/\alpha_{min}}$) and $\alpha_{max} = \ln p_i / \ln l_{-\infty}$ (hence $l_{-\infty} \propto B^{-1/\alpha_{max}}$). Further, the smallest splitting between two nearby orbits, which is of the order of $l_{+\infty}$, becomes at most a splitting of the order of $l_{-\infty}$. With these scaling relations Eq. 4 reads:

$$l_{-\infty}/l_{+\infty} \propto B^{1/(1-q)}, \quad (5)$$

which implies a precise relation between the entropic index q and the extremes of $f(\alpha)$:

$$\frac{1}{1-q} = \frac{1}{\alpha_{min}} - \frac{1}{\alpha_{max}}. \quad (6)$$

The above expression is the main result of the present work. It asserts that, once the scaling properties of the dynamical attractor are known, one can precisely infer about the proper nonextensive statistics that must be used in order to predict the thermodynamics of the system.

Let us illustrate the above result using as prototype multifractal objects the critical attractor of one-dimensional dissipative maps. The scaling properties near the onset of chaos allows for analytical expressions for the end points of the singularity spectrum. As it has been shown by Feigenbaum, with $B = b^n$ -cycle elements on the attractor (b stands for a natural scale for the partitions), the most rarefied and most concentrated elements scale, respectively as $l_{-\infty} \sim \alpha_F^{-n}$ and $l_{+\infty} \sim \alpha_F^{-zn}$, where α_F is the Feigenbaum's universal scaling factor and z represents the nonlinearity (inflexion) of the map at the vicinity of its extremal point [16]. Since the measures there are simply $p_{-\infty} = p_{+\infty} = p_i = b^{-n}$, these end points are respectively

$$\alpha_{max} = \frac{\ln b}{\ln \alpha_F}, \quad (7)$$

$$\alpha_{min} = \frac{\ln b}{z \ln \alpha_F}. \quad (8)$$

Therefore, the entropic index q can be put as a function of the Feigenbaum's scaling factor as

$$\frac{1}{1-q} = (z-1) \frac{\ln \alpha_F}{\ln b}. \quad (9)$$

In order to determine the singularity spectrum of the critical attractor, we implemented the algorithm proposed by Halsey *et al* [14], which has been shown to be more precise than the standard box counting method. Firstly, we consider a family of generalized logistic maps

$$x_{t+1} = 1 - a|x_t|^z \quad (1 < z < \infty; 0 < a \leq 2; -1 \leq x_t \leq 1), \quad (10)$$

which exhibits a period-doubling cascade accumulating at $a_c(z)$ ($b = 2$ is therefore the natural scale for the partitions). Here z is precisely the inflexion of the map at the vicinity of its extremal $\bar{x} = 0$. Typical multifractal singularity spectra are shown in Fig. 1. From their end points we can estimate the z -dependence of the universal scaling factor $\alpha_F(z)$. The so obtained values are shown in figure 2 together with known asymptotics [17] and the parametric dependence of α_{min} and α_{max} on the fractal dimension d_f (see inset). The entropic index q was *independently* obtained (numerically) from the plots of $\ln \xi(N) = \sum_{t=1}^N \ln [az|x_t|^{z-1}]$ versus $\ln N$, where N is the number of iterations. The upper bound of these plots have slopes equals to $1/(1-q)$ (see Fig. 3a). Its fractal-like structure reflects the presence of long-range temporal correlations at the critical point. The so obtained values of q are plotted in Fig. 4 against the numerical values $1/\alpha_{min} - 1/\alpha_{max}$ and corroborate the relation predicted from scaling arguments.

We also computed the multifractal spectrum $f(\alpha)$ and the sensitivity function $\xi(N)$ for the following two-parameter map:

$$x_{t+1} = d \cos(\pi|x_t - 1/2|^{z/2}) \quad (1 < z < \infty; 0 < d < \infty; -d \leq x_t \leq d). \quad (11)$$

This map also display a period doubling route to chaos (z is the inflexion of the map at the vicinity of the extremal point $\bar{x} = 1/2$). A typical onset to chaos is found, for example, to be at $d_c(z = 2) = 0.865579\dots$. Our numerical results confirm, as expected, that this map has, for fixed z , the same $f(\alpha)$ and q of the logistic-like map, i.e., both maps belong to the same universality class.

As a final illustration, we computed $\xi(N)$ for the circle map which is an iterative mapping of one point on a circle to another. This map also exhibits a transition to chaos via quasiperiodic trajectories. It describes dynamical systems possessing a natural frequency ω_1 and driven by an external frequency ω_2 ($\Omega = \omega_1/\omega_2$ is the called *bare* winding number) and belongs to the same universality class of the forced Rayleigh-Bénard convection [18]. These systems tend to mode-lock at a frequency ω_1^* (the ratio between the response frequency to the driving frequency $\omega^* = \omega_1^*/\omega_2$ is usually called *dressed* winding number). The standard one-dimensional version of the circle map reads:

$$\theta_{t+1} = \theta_t + \Omega - \frac{K}{2\pi} \sin(2\pi\theta_t), \quad (0 < \Omega < 1; 0 < K < \infty; 0 < \theta_t < 1). \quad (12)$$

This map exhibits a nonlinear inflexion around its extremal point only for $K > 1$. Therefore $K = 1$ is the onset value above which chaotic orbits exist (for $K < 1$ the orbits are always periodic). A well-studied transition takes place at $K = 1$ and dressed winding number equal to the golden mean $\omega^* = (\sqrt{5} - 1)/2 = 0.61803\dots$, which corresponds to $\Omega = 0.606661\dots$. With these parameters, the map has a cubic inflexion ($z = 3$) near its extremal point $\bar{\theta} = 0$ and its universal scaling factor is found to be $\alpha_F = 1.2885\dots$ [19] ($b = 1/\omega^*$ is the natural scale for the partitions). From Eq. 9 the predicted value for the entropic index is $q = 0.0507\dots$. This value also agrees with the numerical estimation based on the sensitivity to initial conditions (see Fig. 3b).

In conclusion, we have shown how the proper nonextensive statistics can be inferred from the scaling properties of the dynamical attractor at the onset of chaos in one-dimensional dissipative maps. The relation between the entropic index q of generalized statistics and the multifractal singularity spectrum of the dynamical attractor was derived using quite general scaling arguments applied to the most concentrated and rarefied regions of the attractor. The proposed relation is therefore expected to hold for higher dimensional dissipative systems and to provide a close relationship between the nonextensive statistics formalism and the self-organized critical states of large driven dynamical systems. Analogous connections might exist for Hamiltonian systems with long-range interactions.

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I. FIGURE CAPTIONS

Figure 1 - Multifractal singularity spectra of the critical attractor of generalized logistic maps with $z = 1.5, 2.0$ and 3.0 as numerically obtained following the prescription in Ref. [14].

Figure 2 - The logistic-like-map values of $\alpha_F(z)$ (numerically obtained from both α_{min} and α_{max}). The solid lines represent known analytical expressions for the asymptotic behaviors ($[\alpha_F(z)]^z \rightarrow 1/0.033381\dots$ as $z \rightarrow \infty$; and $\alpha_F(z) \sim -1/[(z-1)\ln(z-1)]$ as $z \rightarrow 1$ [17]). [Inset: α_{min} and α_{max} versus d_f . Dashed lines are to guide the eyes.

Figure 3 - $\ln \xi(N)$ versus $\ln N$. (a): standard logistic map ($z=2$). The solid line represents the theoretically predicted slope for the upper bounds $1/(1-q) = \ln \alpha_F(2)/\ln 2 = 1.3236\dots$ ($\alpha_F(2) = 2.5029\dots$ [16]), hence $q = 0.2445\dots$ (b): circle map at $K = 1$ and $\omega^* = (\sqrt{5}-1)/2$ (hence $\Omega = 0.606661\dots$). The solid line represents the theoretically predicted slope for the upper bounds $1/(1-q) = 2 \ln \alpha_F / \ln \omega^* = 1.0534\dots$ ($\alpha_F = 1.2885\dots$ [19]), hence $q = 0.0507\dots$

Figure 4 - q versus $1/\alpha_{min} - 1/\alpha_{max}$ for the generalized logistic map(circles) and for the circle map (square). The straight line represents the scaling prediction.







